

PLETHYSM AND LATTICE POINT COUNTING

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ABSTRACT. We apply lattice point counting methods to compute the multiplicities in the plethysm of $GL(n)$. Our approach gives insight into the asymptotic growth of the plethysm and makes the problem amenable to computer algebra. We prove an old conjecture of Howe on the leading term of plethysm. For any partition μ of 3, 4, or 5 we obtain an explicit formula in λ and k for the multiplicity of S^λ in $S^\mu(S^k)$.

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1. INTRODUCTION

The *plethysm* problem can be stated in different ways. One is to describe the homogeneous polynomials on the spaces $S^k W^*$ and $\bigwedge^k W^*$ in terms of representations of the group $GL(W)$. This is equivalent to decomposing $S^d(S^k W)$ into isotypic components and finding the multiplicity of each isotypic component. The general goal in plethysm is to determine the coefficients of S^λ in $S^\mu(S^\nu W)$ as a function of the partitions λ, μ , and ν . The term plethysm was coined by Littlewood [Lit36], and this type of problems appears in many branches of mathematics beyond representation theory (consult [LR11] for some recent developments in *plethystic calculus*). A general explicit solution of plethysm may be intractable as the resulting formulas are simply too complicated. Here we show piecewise quasi-polynomial formulas that describe the plethysm and then focus on two directions. One is explicit descriptions for small μ which we find with the help of computer algebra. The other direction is asymptotics of plethysm where we confirm a conjecture of Howe [How87, 3.6(d)] on the lead term (Theorem 4.2).

Our contributions are summarized in the following theorem. The proof of the formula is complete after Section 3, while the asymptotics is dealt with in Section 4.

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Theorem 1.1. *Let μ be a fixed partition, k a natural number, and let λ be a partition of $k|\mu|$. The multiplicity of the isotypic component of $S^\mu(S^k W)$ corresponding to λ , as a function of λ and k , is the following piecewise quasi-polynomial:*

$$\frac{\dim \mu}{|\mu|!} \#P_{k,|\mu|}^\lambda + (-1)^{\binom{|\mu|-1}{2}} \left(\sum_{\alpha \vdash |\mu|, \alpha \neq (1, \dots, 1)} \chi_\mu(\alpha) \frac{D_\alpha}{|\mu|!} \sum_{\pi \in S_{|\mu|-1}} \operatorname{sgn}(\pi) Q_\alpha(k, \lambda_\pi) \right),$$

where $P_{k,|\mu|}^\lambda$ is an explicit polytope, χ_μ is the character of the symmetric group $S_{|\mu|}$ corresponding to the partition μ , the Q_α are counting functions for the fibers of projections of explicit polyhedral cones, and λ_π is a linear shift of λ . Moreover, $\frac{\dim \mu}{|\mu|!} \#P_{k,|\mu|}^\lambda$ is the leading term and can be interpreted as coming from the Littlewood–Richardson rule. When μ is any partition of 3, 4, or 5, the explicit piecewise quasi-polynomials have been computed and can be downloaded from the project homepage in ISL format.

Theorem 1.1 yields explicit formulas for plethysm that generalize known results in the cases $|\mu| = 2, 3, 4$. Although these formulas are not necessarily practical to work with on paper, computers are quick to evaluate them, study their asymptotics, and generally extract different sorts of information from them. In this sense, Theorem 1.1 is more effective (but maybe less instructive) than approaches by tableaux counting such as [Rus14]. Although its individual constituents are quasi-polynomials whose chambers are cones, this cannot be guaranteed for the whole expression solely from the formula in Theorem 1.1. We discuss this in Remarks 3.11 and 3.12.

To arrive at the theorem, we first compute the character of the representation $S^\mu(S^k W)$. Using known formulas relating Schur polynomials and complete symmetric polynomials, we relate the multiplicities of isotypic components of the plethysm to coefficients of monomials of a specific polynomial (Propositions 2.8 and 3.4, Section 3.4). We then reduce the determination of these coefficients to a purely combinatorial problem: lattice point counting in certain rational polytopes related to transportation polytopes (Definition 3.6). For fixed μ , the final multiplicity is a function of $\lambda_1, \dots, \lambda_{|\mu|}$ and k . These arguments may belong to a finite number of polyhedral chambers. In each chamber, the result is a quasi-polynomial, that is, a polynomial with coefficients that depend on the remainders of its arguments modulo a fixed number. Equivalently, it is a polynomial in floor functions of linear expressions in the arguments. Software to determine piecewise quasi-polynomials is well-developed due to applications ranging from toric geometry to loop optimization in compiler research. We show how to use BARVINOK [VSB⁺07] and the ISL-library [Ver10] to make Theorem 1.1 explicit. This yields a concrete decomposition of $S^\mu(S^k W)$ (and $S^\mu(\bigwedge^k W)$) for any partition μ of 3, 4, or 5. For each fixed μ , the result is a decomposition of (λ, k) -space into polyhedral chambers, such that in each chamber the multiplicity is a quasi-polynomial. We have set up a homepage for the results in this paper at

<http://www.thomas-kahle.de/plethysm.html>

In the appendix (Section 5) we detail our experiences with the software. Our computations have been carried out with version 0.37 of BARVINOK and version 0.13 of ISL.

Before presenting our methods, we now give a short overview of applications of our results as well as different approaches.

Classical results: Our results extend classical theory. For example, the description of quadrics on the space $S^k(W^*)$ is a classical result of Thrall [Thr42], [CGR84, 4.1–4.6].

Example 1.2. One has $GL(W)$ -module decompositions

$$S^2(S^k W) = \bigoplus S^\lambda W, \quad \Lambda^2(S^k W) = \bigoplus S^\delta W,$$

where the first sum runs over representations corresponding partitions λ of $2k$ into two even parts and the second sum runs over representations corresponding to partitions δ of $2k$ into two odd parts.

The decomposition of cubics $S^3(S^k W)$ is also known. Stated in different forms, it can be found in [Thr42, Plu72, CGR84, How87, Aga02]. In fact, the latter four have formulas for $S^\mu(S^k W)$ for any partition μ of 3. The determination of $S^4(S^k)$ has been addressed in [Fou54, Dun52, How87].

Asymptotics: The explicit formulas for plethysm become complicated quickly, but there is hope for simpler asymptotic formulas. For instance, the decomposition of $S^d(S^k W)$ is related to $(S^k W)^{\otimes d}$ by means of the symmetrizing operator $(S^k W)^{\otimes d} \rightarrow S^d(S^k W)$. There the decomposition of the domain of the resulting quasi-polynomial is known from Pieri's (or more generally the Littlewood–Richardson) rule. In the same vein, Howe [How87, 3.6(d)] identified the leading term for $S^3(S^k)$ and $S^4(S^k)$. A different approach by Fulger and Zhou [FZ15] studies the asymptotics of plethysm by considering how many different irreducible representations and which sums of multiplicities can appear. Further asymptotic results, e.g., when the inner Schur functor is fixed, are presented in [CDKW14]. They are achieved through a connection to the commutation of quantization and reduction [Sja95, MS99, Mei96].

Knowledge of explicit quasi-polynomial formulas allows one to test techniques for studying the asymptotics algorithmically on nontrivial examples. Another insight from Section 4 is that the language of convex discrete geometry may be more useful for proofs than that of piecewise quasi-polynomials.

Evaluation: One of the principal uses of our results is evaluation of the plethysm function. While evaluation for individual values can be done in `LiE` [vLCL92] and other packages, our results are more flexible as they are given as functions on parameter space and can thus be evaluated parametrically.

Example 1.3. Let $\mu = (5)$, $\lambda = (31, 3, 2, 2, 2)$, and make the following definitions:

$$\begin{aligned} p_1 &= -\frac{289}{720}s + \frac{1}{20}s^2 + \frac{1}{720}s^3 \\ p_2 &= \frac{5}{8} + \frac{1}{8}s, \quad p_3 = \frac{1}{3} - \frac{1}{6}s, \quad p_4 = \frac{7}{12} - \frac{1}{3}s, \\ A(s) &= p_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left(p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left\lfloor \frac{1+s}{3} \right\rfloor + \frac{1}{4} \left(\left\lfloor \frac{1+s}{3} \right\rfloor^2 + \left\lfloor \frac{s}{4} \right\rfloor - \left\lfloor \frac{3+s}{4} \right\rfloor \right) \end{aligned}$$

With these definitions the coefficient of $S^{s\lambda}$ in $S^\mu(S^{8s})$ equals

$$A(s) + \begin{cases} 1 & \text{if } s \equiv 0 \pmod{5} \\ \frac{3}{5} & \text{if } s \equiv 1 \pmod{5} \\ \frac{4}{5} & \text{if } s \equiv 2, 3, 4 \pmod{5}, \end{cases}$$

Note that after the reductions in Section 3, Lemma 4.1 with $l = 2, d = 5$ gives a degree bound of three, which is realized here. Also note that the result is a single quasi-polynomial (see Remark 3.11).

Remark 1.4. It can be slightly tricky to automatically extract these kinds of formulas from our computational results. In general, ISCC has somewhat limited capabilities in producing human-readable output. In Example 5.6 we give a complete discussion of how to derive this result using our computational results and ISCC.

Example 1.5. Actual numerical evaluation is very quick too. For instance, the multiplicity of the isotypic component of

$$\lambda = (616036908677580244, 1234567812345678, 12345671234567, 123456123456)$$

in $S^5(S^{123456789123456789})$ equals

$$24096357040623527797673915801061590529381724384546352415930440743659968070016051.$$

The evaluation of our formula on this example takes under one second, and this time is almost entirely constant overhead for dealing with the data structure. Evaluation on much larger arguments (for instance with a million digits) is almost as quick.

Quantum physics: Descriptions of entangled quantum states of bosons and fermions are related to plethysms of symmetric and wedge product (see [CDKW14, CDW12]).

Testing conjectures: Although many theoretical formulas for plethysm are known, some basic properties are mysterious. For example, a conjecture of Foulkes states that for $a < b$, $S^a(S^b)$ embeds as a subrepresentation into $S^b(S^a)$. For $a = 2$, this is a classical result. For $a = 3$, it was shown in this century [DS00] and for $a = 4$ in [McK08]. A variant has been studied in [AC07]. Using explicit quasi-polynomials, one can attack this problem for any fixed a . One would need to compare two explicit quasi-polynomials and in particular decide whether their difference is a positive quasi-polynomial. At the moment, we cannot complete this direction as our methods only work for fixed exponent of the outer Schur functor. Results of Bedratyuk [Bed11] indicate that explicit quasi-polynomials can also be found for fixed exponent of the inner Schur functor, once we fix the group (to be $SL(n)$). Our result can also be considered as a step toward Stanley’s Problem 9 in [Sta00] asking for the combinatorial description of plethysm. For this major breakthrough one would need “positive” formulas, though.

The zero locus of plethysm coefficients: The question of which isotypic components appear in plethysm is highly nontrivial. Some very special cases follow from the resolution of Weintraub’s conjecture [Wei90, BCI11, MM12]. We hope that our formulas can contribute to finding further regularities among partitions that appear in different plethysms, for instance, by studying the zeros of our quasi-polynomials.

Geometric Complexity Theory: The problem of separation of complexity classes is addressed with geometric methods in [MS01, BLMW11]. The crucial point of this program requires comparing closures of orbits of explicit symmetric tensors. The plethysm plays an important role there [MS01, p. 516], [BLMW11, p. 10], [Lan15, p. 10].

Computation of syzygies: The computations of syzygies of homogeneous varieties is related to (inner) plethysm (see [Wey03, p. 63]). Weyman [Wey03, p. 241] applies the explicit computation of plethysm $S^n(S^2)$ to study rank varieties [Wey03, Section 7.1] and topics related to free resolution of the Grassmannian.

Unification: Many specialized methods have been developed to attack plethysm problems, and most of them work only in a very restricted set of exponents. For instance, we have already mentioned several methods to compute $S^3(S^k)$ [Thr42, Plu72, CGR84, How87, Aga02].

Quasi-polynomials and convex bodies provide a unifying framework for all of those techniques. Specializing to certain situations is just partially evaluating the quasi-polynomial. In the history of plethysm, people have written a new paper with a new technique whenever the next exponent was due. Our method, in contrast, stays the same.

Representations of S_n : The plethysm is also related to representations of the permutation group S_n . For two representations of S_a and S_b corresponding to λ and μ , respectively, there is a wreath product representation $\lambda \wr \mu$ of the wreath product group $S_a \wr S_b$. One has a natural inclusion $S_a \wr S_b \subset S_{ab}$. By the Frobenius characteristic map, we can identify representations of symmetric groups with symmetric polynomials, and under this identification, the representation of S_{ab} induced from $\lambda \wr \mu$ is exactly the plethysm of the representations given by λ and μ . The interested reader may consult [Sta00, Vol. 2, Theorem A2.6] and references there. For similar results on the Kronecker coefficients, appearing in the decomposition of the tensor product of representations of symmetric groups, see [BOR09a, BOR11].

Classical algebraic geometry: The spaces $S^k W^*$ and $\bigwedge^k W^*$ are ambient spaces of the Veronese variety and the Grassmannian. These varieties and related objects, e.g., their secant and tangential varieties, have been studied classically (see [Zak93] and references therein). The description of the algebra of the Veronese and Grassmannian is well-known. However, as the decomposition of $S^d(S^k W)$ is not known, the decomposition of the degree d part of the ideal is a difficult problem—even for quartics! Our results provide such a description. Furthermore, due to problems motivated by determining ranks of tensors, secant varieties are often studied from a computational point of view. It is an open problem to check, whether the ideal of the secant (line) variety of any Grassmannian is generated by cubics. A description of all cubics in the ideal was given in [MM15]. It is natural to ask which quadrics are generated by cubics. To answer this question, the description of all degree four polynomials is helpful. Thus, our results provide very practical information. One could argue that we provide the decomposition of very low degree equations. Note, however, that on the k -th secant variety, no equations of degree less than or equal to k vanish. On the other hand, the equations of degree $k+1$ sometimes already provide all generators of the ideal (e.g., the Segre-Veronese varieties for $k = 2$ [Rai12]). Thus, knowing their decompositions is an important first step in determining the structure of the whole ideal. The same method can be applied to other ideals defined by objects related to representation theory. One example is the ideal of relations among $k \times k$ minors of a generic matrix studied in [BCV13].

Errors: As the formulas and computations become more and more technically involved, the chance of human error rises. To quote from Howe [How87]: *Here we will outline what is involved in the computations and list our answers. The details are available from the author on request. The author does hope someone will check the calculations, because he does not have a great deal of faith in his ability to carry through the details in a fault-free manner. He hopes, however, that the answers are qualitatively correct as stated.* We have not checked all historical formulas that overlap with our results, but errors have been identified before (compare [MM15, Appendix] and [CGR84]).

Convention and notation. All representations considered are finite dimensional. Let $\lambda = (\lambda_1, \dots, \lambda_l)$ with $\lambda_1 \geq \dots \geq \lambda_l > 0$ be an integral partition of n , i.e., $\sum_{i=1}^l \lambda_i = n$. We set $|\lambda| = n$. Consider a vector space W of dimension at least l . Let $S^\lambda W$ be the irreducible representation of $GL(W)$ corresponding to λ , obtained by acting with the Young symmetrizer

c_λ on $W^{\otimes n}$ [FH91]. We use the convention that the partition $(1, \dots, 1)$ corresponds to the wedge product representation $\bigwedge^n W$ and (λ_1) to the symmetric power $S^n W$. All irreducible representations of $SL(W)$ can be obtained by considering partitions λ with n arbitrary and $l < \dim W$. For $GL(W)$ the theory is similar. There we have to specify an additional integer r . The vector space and the group action are the same as those for $SL(W)$, but additionally we multiply a given vector by the determinant to the power r .

In general, the *Schur functors* S^λ are endofunctors of the category of representations. Applying them to the standard representation W yields all irreducible representations. Applying S^λ to other irreducible representations, in general, yields reducible representations. The *plethysm* is to understand the decomposition of $S^\lambda(S^\mu W)$.

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2. CHARACTERS

The trace of a $GL(W)$ representation is a symmetric polynomial in the eigenvalues, known as the *character* of the representation. To determine the character, one considers the action of diagonal matrices.

Definition 2.1 ($h_k(x^a)$, ψ_α). Consider d variables x_1, \dots, x_d . For $a \in \mathbb{N}$, let $h_k(x^a)$ be the complete symmetric polynomial of degree k in the variables x_1^a, \dots, x_d^a . Let α be a multi-index of length j . We define the polynomials

$$\psi_n = \sum_i x_i^n, \quad \psi_\alpha = \prod_{i=1}^j \psi_{\alpha_i}, \quad \text{and} \quad \psi_\alpha \circ h_k = \prod_{i=1}^j h_k(x^{\alpha_i}).$$

Example 2.2. The character of the representation $S^k(W)$ is the sum of all monomials of degree k in $\dim W$ variables, that is $h_k(x)$. For $\bigwedge^k W$, we obtain the sum of all square-free monomials of degree k , known as the elementary symmetric polynomial.

For any representation V , the associated character is denoted by P_V . The character of the irreducible representation $S^\lambda W$ is the *Schur polynomial* P_λ . Schur polynomials are independent and form a basis of symmetric polynomials. Since the character of a sum of two representations is the sum of their characters, in order to decompose any representation, it is enough to express its character as a sum of Schur polynomials. Precisely, $V = \sum (S^\lambda V)^{\oplus a_\lambda}$ if and only if $P_V = \sum a_\lambda P_\lambda$. Other operations on representations translate too. For instance, the plethysm of two symmetric polynomials f, g is the composition $f \circ g$ (see [Mac98, I.8] for a precise algebraic definition).

Proposition 2.3 ([Mac98, I.8.3, I.8.4, I.8.6]). *For any symmetric polynomial f , the mapping $g \rightarrow g \circ f$ is an endomorphism of the ring of symmetric polynomials. For any $n \in \mathbb{N}$, the*

mapping $g \rightarrow \psi_n \circ g$ is an endomorphism of the ring of symmetric polynomials. Moreover,

$$\psi_n \circ g = g \circ \psi_n = g(x_1^n, x_2^n, \dots).$$

Remark 2.4. Proposition 2.3 justifies the notation

$$\psi_\alpha \circ h_k = \prod_{i=1}^j h_k(x^{\alpha_i}).$$

From now on, assume that $\dim W$ is large enough so that all appearing partitions have at most $\dim W$ parts and fix a partition μ of an integer d . Irreducible representations of the permutation group S_d are indexed by Young diagrams with exactly d boxes. The character corresponding to the Young diagram $\rho \vdash d$ is denoted χ_ρ .

Definition 2.5 (z_ρ , [Mac98, p.17]). Let $\rho = (\rho_1 \geq \dots \geq \rho_k)$ be a partition of d and m_i the number of parts equal to i . We define

$$z_\rho = \prod_{i \geq 1} i^{m_i} m_i! = \frac{d!}{D_\rho},$$

where D_ρ is the number of permutations of cycle type ρ .

Remark 2.6. With [Mac98, I.7.(7.2) and I.7.(7.5)] the character P_μ can be expressed in terms of ψ_n as

$$P_\mu = \sum_{\rho \vdash d} z_\rho^{-1} \chi_\mu(\rho) \psi_\rho,$$

where $\chi_\mu(\rho)$ is the value of the character χ_μ on (any) permutation of type ρ .

Example 2.7. As the Young diagram (d) corresponds to the trivial representation of S_d , we obtain the formula for the complete symmetric polynomial:

$$h_d = P_{(d)} = \sum_{\rho \vdash d} \frac{D_\rho}{d!} \psi_\rho,$$

where D_ρ is the number of permutations of combinatorial type ρ in the group S_d . The Young diagram $(1, \dots, 1)$ corresponds to the sign representation of S_d . Hence, we obtain the formula for the character of the wedge power:

$$P_{(1, \dots, 1)} = \sum_{\rho \vdash d} \text{sgn}(\rho) \frac{D_\rho}{d!} \psi_\rho.$$

Proposition 2.8. The character of the representation $S^\mu(S^k W)$ equals

$$P_{S^\mu(S^k W)} = \sum_{\alpha} \chi_\mu(\alpha) \frac{D_\alpha}{d!} \psi_\alpha \circ h_k,$$

where the sum is taken over all partitions α of $d := |\mu|$ and D_α is the number of permutations of cycle type α in the group S_d .

Proof. We have

$$P_{S^\mu(S^k W)} = P_{S^\mu} \circ h_k.$$

By Remark 2.6, this equals

$$\sum_{\rho \vdash d} z_\rho^{-1} \chi_\mu(\rho) \psi_\rho \circ h_k. \quad \square$$

Remark 2.9. A similar formula for arbitrary composition of Schur functors is presented in [Yan98, Theorem 2.2]. We do not apply it directly, as it relies on ‘nested inverse Kostka numbers’. As explained in [Yan98, Yan02], the computation of those, although possible in many cases, is a nontrivial task. For this reason, we introduce one more change of basis of symmetric polynomials, relating our results to transportation polytopes. From the algorithmic point of view, although the final result counts the same multiplicities, enumeration of points in dilated polytopes is easier than enumeration of skew Young diagrams with specific properties.

For fixed d , all partitions can be listed and the decomposition of $P_{S^\mu(S^k W)}$ into Schur polynomials reduces to the decomposition of each polynomial $\psi_\alpha \circ h_k$. Indeed, the values of $\chi_\mu(\rho)$ can be made explicit by the celebrated Frobenius Formula [FH91, 4.10]. As similar results will be used later, we review the formula in detail.

Definition 2.10 ($[P]_\alpha, \Delta(x)$). For any polynomial P and partition $\alpha = (\alpha_1, \dots, \alpha_k)$, define $[P]_\alpha$ as the coefficient of the monomial $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ in P .

Definition 2.11. For a fixed number of variables x_1, \dots, x_k , the *discriminant* is

$$\Delta(x) = \prod_{i < j} (x_i - x_j).$$

The value of the character χ_μ on any permutation of cycle type ρ equals:

$$(2.1) \quad \chi_\mu(\rho) = [\Delta(x)\psi_\rho]_{(\mu_1+k-1, \mu_2+k-2, \dots, \mu_k)}. \quad (\text{Frobenius formula})$$

Example 2.12. Consider a permutation $\pi \in S_4$ of cycle type $(3, 1)$, e.g., the permutation that fixes 4 and permutes $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$. Consider a representation corresponding to the partition $2 + 2 = 4$. We obtain:

$$\chi_{(2,2)}(\pi) = [(x_1 - x_2)(x_1^3 + x_2^3)(x_1 + x_2)]_{(3,2)} = [x_1^5 - x_1^3 x_2^2 + x_1^2 x_2^3 - x_2^5]_{(3,2)} = -1.$$

3. REDUCTIONS

To make Proposition 2.8 effective, we employ the following simplifications.

- (1) Reduction of the number of variables.
- (2) Application of the Littlewood–Richardson rule to the most complicated term.
- (3) Change of basis of symmetric functions.
- (4) Reduction to combinatorics of polytopes.

3.1. Reduction of the number of variables. Our aim is to compute the multiplicity of the isotypic component corresponding to λ inside $S^\mu(S^k W)$. By the Littlewood–Richardson rule, λ can have at most $|\mu|$ rows, so we can assume $\dim W = |\mu|$.

Proposition 3.1 ([Car90], [Man98]).

$$S^\mu(S^{2l} W) = S^\mu\left(\bigwedge^{2l} W\right)^\vee, \quad S^\mu(S^{2l+1} W) = S^{\mu^\vee}\left(\bigwedge^{2l+1} W\right)^\vee,$$

where $(.)^\vee$ stands for the representation arising from $(.)$ by replacing each irreducible component corresponding to a Young diagram ν with the component corresponding to the transpose of ν , denoted ν^\vee .

Proposition 3.1 says that the multiplicity of an isotypic component corresponding to λ inside $S^\mu(S^l W)$ is the multiplicity of λ^\vee inside either $S^{\mu^\vee}(\bigwedge^l W)$ or $S^\mu(\bigwedge^l W)$. For the wedge power, the following well-known reductions hold which we prove for the sake of completeness.

Lemma 3.2 (Reduction Lemma [Car90, 5.8, 5.9], [MM15, Lemma 6.3]). *Let μ be any Young diagram of weight d , and λ a Young diagram with d columns and weight dk . Let λ' equal λ with the first row removed. The multiplicity of the component corresponding to λ in $S^\mu(\bigwedge^k W)$ equals the multiplicity of the component corresponding to λ' in $S^\mu(\bigwedge^{k-1} W)$.*

Proof. Consider the inclusion $S^\mu(\bigwedge^k W) \subset (\bigwedge^k W)^{\otimes d}$ with a basis given by tensor products of wedge products of basis elements of W . Each vector in the highest weight space corresponding to λ must contain exactly one e_1 in each tensor. We get an isomorphism of highest weight spaces by removing e_1 and decreasing the indices of other basis vectors by one. \square

The above facts show that whenever λ^\vee has d nonzero columns (or equivalently λ has d nonzero rows), we can express the multiplicity in the plethysm by a multiplicity in a simpler plethysm. It follows that it is enough to determine the multiplicities of isotypic components corresponding to λ with at most $d - 1$ rows. This is equivalent to the assumption that $\dim W = d - 1$ or that the symmetric polynomials are in variables x_1, \dots, x_{d-1} . From now on, we make this assumption, recovering the general case at the end (Remark 3.8).

3.2. Application of Littlewood–Richardson rule. Suppose

$$\psi_\alpha \circ h_k = \sum_{\lambda} a_{\alpha, \lambda} S^\lambda,$$

where S^λ is the Schur polynomial corresponding to λ and the sum is over all partitions $\lambda \vdash dk$, with at most $d - 1$ parts. In the following sections, we associate polytopes to the polynomials $\psi_\alpha \circ h_k$. Although our computer algebraic methods work in general, they are least efficient for the partition $\alpha = (1, \dots, 1)$. In this section, we show how to express $\psi_{(1, \dots, 1)} \circ h_k$ in terms of Schur polynomials without further computation. While in the end these reductions were not necessary in our computations, we present them as an introduction to the methods in the remaining sections and to better understand the leading term in the plethysm formula.

Fix $\alpha_0 = (1, \dots, 1) \vdash d$. By Remark 2.4, $\psi_{\alpha_0} \circ h_k = (h_k(x))^d$, the d -th power of the complete symmetric polynomial of degree k . As multiplication of polynomials corresponds to the tensor product of representations, this is the character of the representation $(S^k W)^{\otimes d}$. The decomposition of this representation is known due to Pieri's rule (or more generally the Littlewood–Richardson rule). In order to make the formulas explicit, consider the following polytope.

Definition 3.3 (The polytope $P_{k,d}$). Let $(x_1^1, x_1^2, x_2^2, \dots, x_1^{d-1}, \dots, x_{d-1}^{d-1})$ denote coordinates of the vector space $\mathbb{R}^1 \times \mathbb{R}^2 \times \dots \times \mathbb{R}^{d-1}$. Denote $x_1^0 = k$, $x_{j+1}^j = k - \sum_{i=1}^j x_i^j$ and $x_i^j = 0$ for $i > j + 1$. Let $P_{k,d}$ be the polytope defined by the following constraints:

- (1) $x_i^j \geq 0$, for all i, j ,
- (2) $\sum_{l \leq j} x_i^l \leq \sum_{l \leq j-1} x_{i-1}^l$, for all j and $1 < i \leq j + 1$.

In Definition 3.3, x_i^j corresponds to the number of boxes added according to Pieri's rule in the j -th step in the i -th row. For a polytope P , let $\#P$ denote the number of integral points in P . By Pieri's rule we obtain the following

Proposition 3.4. *The coefficient $a_{\alpha_0, \lambda}$ in the expansion*

$$\psi_{\alpha_0} \circ h_k = \sum_{\lambda} a_{\alpha_0, \lambda} S^{\lambda},$$

equals the number of integral points in $P_{k,d}$ intersected with the hyperplanes $\sum_j x_i^j = \lambda_i$. In particular, it can be computed as the number of points in the fiber of a projection of $P_{k,d}$. We will denote the intersection by $P_{k,d}^{\lambda}$. \square

Remark 3.5. There are other methods to compute the Littlewood–Richardson coefficients, e.g., due to Berenstein and Zelevinsky [BZ92], that could provide other polytopal descriptions. Contrary to plethysm, the question which representations S^{ν} appear (with positive multiplicities) in $S^{\lambda} \otimes S^{\mu}$ is well-understood [Kly98, KT99, KTW04].

3.3. Change of basis. Suppose

$$\psi_{\alpha} \circ h_k = \sum_{\lambda} a_{\alpha, \lambda} S^{\lambda},$$

where S^{λ} is the Schur polynomial corresponding to λ and the sum is taken over all partitions $\lambda \vdash dk$, with at most $d-1$ parts. By the results of [FH91, Appendix A] and [Mac98], the coefficient $a_{\alpha, \lambda}$ is equal to the coefficient of the monomial $x_1^{\lambda_1+d-2} \cdots x_{d-1}^{\lambda_{d-1}}$ in the polynomial $(\psi_{\alpha} \circ h_k) \prod_{i < j} (x_i - x_j)$, that is:

$$a_{\alpha, \lambda} = [\Delta(x)(\psi_{\alpha} \circ h_k)]_{(\lambda_1+d-2, \lambda_2+d-3, \dots, \lambda_{d-1})}$$

3.4. Integral points in polytopes. When d is fixed, the discriminant $\prod_{i < j} (x_i - x_j)$ is explicit. Our aim is to compute the coefficients of the monomials appearing in $\Delta(x)(\psi_{\alpha} \circ h_k)$.

Definition 3.6 ((α, λ) -matrix). Fix partitions α, λ and suppose that α has a parts. An $a \times (d-1)$ matrix M with nonnegative integral entries is an (α, λ) -matrix if

- (1) each row sums up to k , i.e., $\sum_{j=1}^{d-1} M_{i,j} = k$ for each $1 \leq i \leq a$, and
- (2) the α -weighted entries of the j -th column sum up to λ_j , i.e., $\sum_{i=1}^a \alpha_i M_{i,j} = \lambda_j$ for each $1 \leq j \leq d-1$.

Example 3.7. Let $d = 3$ and $\alpha = (3)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. According to Definition 3.6, an (α, λ) -matrix is a nonnegative integral (1×2) matrix $M = (M_{11}, M_{12})$ satisfying $M_{11} + M_{12} = k$ and $3M = (\lambda_1, \lambda_2)$. There is no such matrix unless $\lambda_1 \equiv 0 \pmod{3}$, and if this is the case, for each k , there is exactly one such matrix if and only if $\lambda_2 = 3k - \lambda_1$.

It is a straightforward observation that the coefficient of x^{λ} in $\psi_{\alpha} \circ h_k$ equals the number of different (α, λ) -matrices, as each matrix encodes the expansion of the product $\prod_{i=1}^a h_k(x^{\alpha_i})$. We want to obtain an explicit formula for the number of (α, λ) -matrices for fixed α as a piecewise quasi-polynomial in $k, \lambda_1, \dots, \lambda_{d-2}$ (λ_{d-1} is determined as $\sum_{i=1}^{d-1} \lambda_i = kd$). Denote this quasi-polynomial by Q_{α} such that

$$\psi_{\alpha} \circ h_k = \sum_{\lambda} Q_{\alpha}(k, \lambda_1, \dots, \lambda_{d-2}) x^{\lambda}.$$

Hence, by the Vandermonde formula,

$$\begin{aligned} \psi_\alpha \circ h_k \prod_{i < j} (x_i - x_j) &= \psi_\alpha \circ h_k (-1)^{\binom{d-1}{2}} \prod_{i < j} (x_j - x_i) \\ &= (-1)^{\binom{d-1}{2}} \left(\sum_{\lambda} Q_\alpha(k, \lambda_1, \dots, \lambda_{d-2}) x^\lambda \right) \left(\sum_{\pi \in S_{d-1}} \operatorname{sgn}(\pi) \prod_{i=1}^{d-1} x_i^{\pi(i)-1} \right). \end{aligned}$$

Consequently the coefficient of $x_1^{\lambda_1+d-2} \cdots x_{d-1}^{\lambda_{d-1}}$ in $(\psi_\alpha \circ h_k) \prod_{i < j} (x_i - x_j)$ equals:

$$(-1)^{\binom{d-1}{2}} \sum_{\pi \in S_{d-1}} \operatorname{sgn}(\pi) Q_\alpha(k, \lambda_1 + d - 1 - \pi(1), \lambda_2 + d - 2 - \pi(2), \dots, \lambda_{d-2} + 2 - \pi(d-2)).$$

For each permutation $\pi \in S_{d-1}$, denote $\lambda_\pi = (\lambda_1 + d - 1 - \pi(1), \lambda_2 + d - 2 - \pi(2), \dots, \lambda_{d-2} + 2 - \pi(d-2))$. Using this notation we obtain the formula for the multiplicity a_λ of the isotypic component corresponding to λ inside $S^\mu(S^k W)$ for μ a partition of d :

$$(-1)^{\binom{d-1}{2}} \left(\sum_{\alpha \vdash d} \chi_\mu(\alpha) \frac{D_\alpha}{d!} \sum_{\pi \in S_{d-1}} \operatorname{sgn}(\pi) Q_\alpha(k, \lambda_\pi) \right).$$

The summand for the partition $\alpha = (1, \dots, 1)$ can be made explicit:

$$(3.1) \quad \frac{\dim \mu}{d!} \#P_{k, |\mu|}^\lambda + (-1)^{\binom{d-1}{2}} \left(\sum_{\alpha \vdash d, \alpha \neq (1, \dots, 1)} \chi_\mu(\alpha) \frac{D_\alpha}{d!} \sum_{\pi \in S_{d-1}} \operatorname{sgn}(\pi) Q_\alpha(k, \lambda_\pi) \right),$$

where $\dim \mu = \chi_\mu(1, \dots, 1)$ is the value of the character χ_μ on the trivial permutation and thus equal to the dimension of the representation of $S_{|\mu|}$ corresponding to μ . We may identify S_{d-1} with the Weyl group \mathcal{W} . Let ρ be half of the sum of positive weights. For esthetic reasons, we may rewrite the above formulas as follows

$$(-1)^{\binom{d-1}{2}} \left(\sum_{\alpha \vdash d} \chi_\mu(\alpha) \frac{D_\alpha}{d!} \sum_{\pi \in \mathcal{W}} \operatorname{sgn}(\pi) Q_\alpha(k, \lambda + \rho - \pi(\rho)) \right).$$

All together, we have reduced the problem of finding the coefficients of the plethysm to computing the piecewise quasi-polynomials Q_α that count the number of (α, λ) -matrices. Let α be a partition with a parts. The integral $(a \times (d-1))$ -matrices form an $a(d-1)$ -dimensional lattice and the linear equations in Definition 3.6 define hyperplanes in this lattice. When L denotes the resulting affine sublattice, the (α, λ) -matrices are simply the nonnegative integer points in L . Alternatively, let $P_{\alpha, \lambda}$ (not to be confused with $P_{k, d}$) be the (rational) polytope $(L \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathbb{Q}_{\geq 0}^{a(d-1)}$. It is a polytope since each coordinate is nonnegative and bounded from above by $\max \lambda_i$. The number of (α, λ) -matrices equals $\#P_{\alpha, \lambda}$, the number of integral points in $P_{\alpha, \lambda}$. It is also worth noting that for any partition α , the polytope $P_{\alpha, \lambda}$ can be obtained from the $P_{(1, \dots, 1), \lambda}$ by a series of hyperplane cuts given by equalities of coordinates. The polytopes $P_{(1, \dots, 1), \lambda}$ are *transportation polytopes*, well studied objects in combinatorics and optimization [KW68, Bol72, BR93, DLK14, Liu13].

The following remark follows by combining Proposition 3.1 and Lemma 3.2.

Remark 3.8. Let μ be a partition of d and $\lambda = (\lambda_1, \dots, \lambda_d)$ with $\sum \lambda_i = dk$. The multiplicity of λ in $S^\mu(S^k)$ equals

- (1) the multiplicity of $(\lambda_1 - \lambda_d, \dots, \lambda_{d-1} - \lambda_d, 0)$ in $S^\mu(S^{k-\lambda_d})$ if λ_d is even,

- (2) the multiplicity of $(\lambda_1 - \lambda_d, \dots, \lambda_{d-1} - \lambda_d, 0)$ in $S^{\mu^\vee}(S^{k-\lambda_d})$ if λ_d is odd, where μ^\vee is the transpose of μ .

Additionally, the value $\lambda_1 - \lambda_d$ is determined by the equation $d(k - \lambda_d) = \sum_{i=1}^{d-1} \lambda_i - (d-1)\lambda_d$. Consequently, our implementation uses arguments

$$(b_1, \dots, b_{d-2}, s) = (\lambda_{d-1} - \lambda_d, \dots, \lambda_2 - \lambda_d, k - \lambda_d).$$

Remark 3.9 (Stable multiplicities). Fix an integer d and let λ be a Young tableau. For every sufficiently large k , we can construct another tableau $\lambda'(k)$ by adding a new first row to λ such that $|\lambda'(k)| = dk$. As a function of k , the multiplicity of the isotypic component $\lambda'(k)$ in $S^d(S^k)$ becomes eventually constant as k grows [Wei90, CT92, Bri93, Man98]. This fact follows easily in our setting. Indeed, note that the desired multiplicity is a function of counts of (α, λ') -matrices. Now when k is very large each possible filling of the columns 2 to a of an (α, λ') -matrix (restricted by the conditions coming from λ) can be uniquely completed.

Remark 3.10. Another possible approach to lattice point counting problems is through Brion-Vergne formula [BV97, p. 802 Theorem (ii)] or [BBCV06] for vector partition functions. It provides an expression for the number of lattice points in polytopes depending on shifts of facets. Our approach here is much more elementary.

Remark 3.11. The lattice point enumerators Q_α have chamber decompositions into polyhedral cones. We believe that the same fact also holds for the whole expression in (3.1). This can not be deduced from the formula directly since shifting the arguments of Q_α by π creates small bounded chambers. This fact also complicates our computations since the software is incapable of unifying chambers, even if the quasi-polynomials on neighboring chambers agree. Once there is theorem that guarantees a chamber decomposition into cones, the computation should be revisited, because then the cones can be computed in advance and the quasi-polynomials can be determined using (3.1). A possible approach to this problem is outlined in Remark 3.12.

Remark 3.12. A very general theory of Meinrenken and Sjamaar on the representation theory of moment bundles on symplectic manifolds may be applicable to the plethysm. More specifically, let M be a symplectic manifold with a Hamiltonian action of a connected compact Lie group G . Let L be a G -equivariant line bundle and denote $RR(M, L)$ the push-forward of L to a point. One may view $RR(M, L)$ either as a complex of representations $H^i(M, L)$ with trivial derivations, or as an element of equivariant K -theory (the representation ring). Now let $N^m(\lambda)$ denote the multiplicity of the representation corresponding to a partition λ in $RR(M, L^{\otimes m})$. Corollary 2.12 in [MS99] says that for every moment bundle L on M the function $N^m(\lambda)$ (as a function of m and λ) is a piecewise quasi-polynomials with closed cones as chambers. In particular, each ray is contained in a single chamber.

To get a result for plethysm, one has to find a suitable manifold M , line bundle L and group G . Results of Brion [Bri93] show how to get the ingredients. A graded module structure on $\sum_k S^\mu(S^{k\nu}V)$ can be obtained as follows. Let X be the affine cone over the unique closed orbit in $\mathbb{P}(S^\nu(V))$, i.e., $X = \text{Spec}(\sum_k S^{k\nu}V)$. Let $T := \{(t_1, \dots, t_{|\mu|}) \in (\mathbb{C}^*)^{|\mu|} : \prod t_i = 1\}$ be the $|\mu| - 1$ dimensional torus. The semidirect product $\Gamma := S_{|\mu|} \ltimes T$ acts on $\mathbb{C}[X]^{\otimes |\mu|} \otimes [\mu]$, where $[\mu]$ is the representation of $S_{|\mu|}$. The invariants of Γ are isomorphic, as a graded module, to $\sum_k S^\mu(S^{k\nu}V)$. Now assume $\nu = (1)$ and $\mu = (|\mu|)$. We obtain a graded algebra structure on $\mathbb{C}[X]^{\otimes |\mu|}$ which in this case is just a tensor power of a polynomial ring with Γ and $C^* \times GL(V)$ actions. In particular, the corresponding variety

is smooth. Since the actions of Γ and $\mathbb{C}^* \times GL(V)$ commute, we may identify the isotypic component corresponding to λ in the plethysm with the invariants $(\mathbb{C}[X]^{\otimes |\mu|} \otimes [\mu])^\Gamma$. Hence, the space of global sections of the line bundle $\mathcal{O}(k)$ on $\text{Proj}(\mathbb{C}[X]^{\otimes |\mu|})$ acquires an additional action of the finite group $S_{|\mu|}$. Meinrenken-Sjamaar's result does not directly apply since the factor $S_{|\mu|}$ makes the group nonconnected. As pointed out to us by Michele Vergne, the theory could be extended to this case. We leave this for future work, but the feasibility has been demonstrated by Manivel, who used the above method to get structural results about the asymptotics of Kronecker coefficients [Man14, Section 2.4].

4. ASYMPTOTIC BEHAVIOR

Our main formula (3.1) also provides insight into the asymptotical properties of plethysm. The main aim here is to identify the leading terms of the piecewise quasi-polynomials that we obtain. As already conjectured by Howe [How87], it is natural to expect that the leading terms come from the polytope of highest dimension, i.e., from the coefficient in the tensor product. This is not obvious since the contribution of a polytope in the quasi-polynomial is *not* of degree equal to the dimension of the polytope. The reason is the signed summations in the formula which decrease the degree. Below we show how to control this type of cancelation, which allows us to obtain the asymptotics. Our strategy is as follows:

- (1) Introduce a new variable s .
- (2) Multiply each variable in the quasi-polynomial by s and ask for the leading term with respect to the degree of s in order to identify the leading term.
- (3) Show that the contribution from polytopes of smaller dimension is strictly smaller than the contribution from the Littlewood–Richardson rule.

More precisely, we compute the multiplicity of $s\lambda$ inside $S^\mu(S^{sk})$ for $s \in \mathbb{N}$ for regular λ , that is, when $\lambda_i \neq \lambda_j$ for all $i \neq j$. In this case, all polytopes appearing in the computation of (3.1) are dilations of $P_{\alpha,\lambda}$ and $P_{k,|\mu|}^\lambda$. The Hilbert–Ehrhart quasi-polynomials of these polytopes are particularly important for us. We can compute the leading term of $\#P_{sk,|\mu|}^{s\lambda}$, which is $\text{Vol } P_{k,|\mu|}^\lambda s^{\dim P_{k,|\mu|}^\lambda}$. One expects this term to be the leading term of the entire formula, as the dimension of $P_{\alpha,\lambda}$ is largest when $\alpha = (1, \dots, 1)$, the Littlewood–Richardson contribution. Indeed, assume that α has a parts and λ has l parts. As we are only interested in partitions $s\lambda$, we can assume that we work with exactly l variables. We have $\dim P_{\alpha,\lambda} = (a-1)(l-1)$. In contrast if $\lambda = (\lambda_1^{a_1}, \dots, \lambda_q^{a_q})$, with $l = \sum_{j=1}^q a_j$, then

$$\dim P_{k,d}^\lambda = 1 + \dots + (l-1) + (l-1)(d-l) - \sum_{j=1}^q \binom{a_j}{2} = (l-1)(d-l/2-1) - \sum_{j=1}^q \binom{a_j}{2}.$$

We omit the easy but tedious proof of this fact as we do not need it below. Note that for regular λ , the dimension equals $(l-1)(d-l/2-1)$. One is tempted to conjecture, as in [How87, 3.6(d)], that the leading term of the multiplicity of the isotypic component corresponding to $s\lambda$ comes from $\#P_{k,d}^\lambda$, as above. It is obvious that this term appears, due to the Littlewood–Richardson rule. The main difficulty in bounding the contributions from the other terms is that the counting function is a piecewise quasi-polynomial: the shifts of argument by the permutation π may change both the chamber and the coefficients of the polynomial. We now provide the estimates for the function $\sum_{\pi \in S_{d-1}} \text{sgn}(\pi) Q_\alpha(k, \lambda_\pi)$.

Lemma 4.1. *Suppose α has $a < d$ parts and λ has l parts. The leading coefficient of*

$$\sum_{\pi \in S_l} \text{sgn}(\pi) Q_\alpha(sk, (s\lambda)_\pi)$$

has degree strictly smaller than $(l-1)(d-l/2-1)$ with respect to the variable s .

Proof. Suppose that α has w parts greater than 1 and h parts equal to 1. In particular, $2w + h \leq d$. Each $(\alpha, (s\lambda)_\pi)$ -matrix M is uniquely determined by two matrices (M_1, M_2) , where M_1 is the $(w \times l)$ -submatrix of M , corresponding to rows with coefficients not equal to one and M_2 the complementary $(h \times l)$ -submatrix. Let α' be the partition of $d-h$ obtained from α by forgetting the singletons, and let $\alpha_0 := (1, \dots, 1) \vdash h$. Introducing parameters i_j for $1 \leq j \leq l-1$ corresponding to column sums of M_1 , we obtain:

$$Q_\alpha(sk, (s\lambda)_\pi) = \sum_{i_1=0}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=0}^{s\lambda_{l-1}+2} Q_{\alpha'}(sk, (i_1 \dots i_{l-1})) Q_{\alpha_0}(sk, (s\lambda)_\pi - (i_1 \dots i_{l-1})).$$

Note that, if $i_j > s\lambda_j + (l+1-j) - \pi(j)$ then $Q_{\alpha_0}(sk, (s\lambda)_\pi - (i_1 \dots i_{l-1})) = 0$, so we could restrict the summation indices; however, we prefer not to. We obtain:

$$\begin{aligned} \sum_{\pi \in S_l} \text{sgn}(\pi) Q_\alpha(sk, (s\lambda)_\pi) = \\ \sum_{i_1=0}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=0}^{s\lambda_{l-1}+2} Q_{\alpha'}(sk, (i_1 \dots i_{l-1})) \left(\sum_{\pi \in S_l} \text{sgn}(\pi) Q_{\alpha_0}(sk, (s\lambda)_\pi - (i_1 \dots i_{l-1})) \right). \end{aligned}$$

We will bound $\left| \sum_{\pi \in S_l} \text{sgn}(\pi) Q_\alpha(sk, (s\lambda)_\pi) \right|$ by a polynomial in s of small degree. To do this, it is enough to bound

$$\sum_{i_1=0}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=0}^{s\lambda_{l-1}+2} Q_{\alpha'}(sk, (i_1 \dots i_{l-1})) \left| \sum_{\pi \in S_l} \text{sgn}(\pi) Q_{\alpha_0}(sk, (s\lambda)_\pi - (i_1 \dots i_{l-1})) \right|.$$

For any sequence of numbers $(\rho_i)_{i=1}^c$ of length c , let $\varsigma \in S_c$ be a permutation that sorts ρ , i.e., $\rho_{\varsigma(i)} \geq \rho_{\varsigma(j)}$, for $i \leq j$. We denote the sorted sequence by $\varsigma(\rho) = (\varsigma_i(\rho))_{i=1}^c$. Let i_l be defined by $\sum_{j=1}^l i_j = sk(d-h)$. A technical problem in the following argument is that the sequence $(s\lambda - (i_1 \dots i_l))$ may be not ordered.

$$\begin{aligned} Q_{\alpha_0}(sk, (s\lambda)_\pi - (i_1 \dots i_l)) &= Q_{\alpha_0}(sk, (s\lambda - (i_1 \dots i_l))_\pi) \\ &= Q_{\alpha_0}\left(sk, (s\lambda_j - i_j + l - (j-1) - \pi(j))_{j=1}^{l-1}\right) \\ &= Q_{\alpha_0}\left(sk, \left(\varsigma_t\left((s\lambda_j - i_j + l - (j-1))_{j=1}^l\right) - \pi(\varsigma(t))\right)_{t=1}^{l-1}\right). \end{aligned}$$

The purpose of this computation was simply to sort the arguments of Q_{α_0} but with the additional complication that the sorting acts on sequences of length l , while only $l-1$ arguments are used by Q_{α_0} . Now, if the sequence $\varsigma\left((s\lambda_j - i_j + l - (j-1))_{j=1}^l\right)$ has two equal entries, then $\sum_{\pi} \text{sgn}(\pi) Q_\alpha(\cdot)$ vanishes because we can match up terms for the permutations differing by the transposition exchanging the two corresponding indices. Note that we use

the symmetry of the counting problem for Q_α too. In this case, the bound holds trivially. Now assume that the sequence is strictly decreasing. We get that

$$(s\lambda_{\varsigma(j)} - i_{\varsigma(j)} + l - (\varsigma(j) - 1) - (l - (j - 1)))_{j=1}^l$$

is nonincreasing. Hence,

$$\left| \sum_{\pi \in S_l} \text{sgn}(\pi) Q_{\alpha_0}(sk, (s\lambda)_\pi - (i_1, \dots, i_{l-1})) \right| = \left| \sum_{\pi \in S_l} \text{sgn}(\pi) Q_{\alpha_0} \left(sk, \left((s\lambda_{\varsigma(j)} - i_{\varsigma(j)} + l - (\varsigma(j) - 1) - (l - (j - 1)))_{j=1}^l \right)_\pi \right) \right|.$$

Now, by the arguments in Sections 3.2 and 3.3 this expression equals the multiplicity of the isotypic component corresponding to the partition

$$(4.1) \quad (s\lambda_{\varsigma(j)} - i_{\varsigma(j)} + l - (\varsigma(j) - 1) - (l - (j - 1)))_{j=1}^l = (s\lambda_{\varsigma(j)} - i_{\varsigma(j)} + j - \varsigma(j))_{j=1}^l$$

inside $(S^{sk})^{\otimes h}$. This allows us to bound the degree with which s may appear separately. The degree of s in the term $Q_{\alpha'}(sk, (i_1 \dots i_{l-1}))$ can be naively bounded by $(w-1)(l-1)$, as each entry of the j -th column of M_1 is bounded by $s\lambda_j$ plus a constant, and the row and column sums of M_1 are fixed. It thus remains to bound the degree of s in

$$(4.2) \quad \sum_{i_1=0}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=0}^{s\lambda_{l-1}+2} \left| \sum_{\pi \in S_l} \text{sgn}(\pi) Q_{\alpha_0} \left(sk, \left((s\lambda_{\varsigma(j)} - i_{\varsigma(j)} + j - \varsigma(j))_{j=1}^l \right)_\pi \right) \right|.$$

In the summation over the indices i_j , there may arise duplicates among the partitions 4.1. However, the number of duplicates of a given weight is bounded by $d!$, since for a fixed ς , there can be at most one sequence $(i_j)_j$ yielding a given partition. For any representation W , let $\mathfrak{s}_l(W)$ be sum of multiplicities of all isotypic components indexed by partitions with at most l parts. We obtain that (4.2) is bounded by $d! \mathfrak{s}_l((S^{sk})^{\otimes h})$. It remains to bound the s -degree of $\mathfrak{s}_l((S^{sk})^{\otimes h})$. We distinguish two cases depending on whether h or l is larger.

Case 1 ($h \geq l$): By Pieri's rule, the multiplicities of isotypic components corresponding to partitions with at most l parts in $(S^{sk})^{\otimes h}$ are determined by the following parameters:

- One parameter for the number of boxes added to the first row in the first step (the remaining boxes going into the second row),
- Two parameters for the number of boxes added to the first and second rows in step 2,
- $i \leq l-1$ parameters for the number of boxes added to rows 1 to i in step i ,
- $(h-l)(l-1)$ parameters for the numbers of boxes added to rows from 1 to $l-1$ in steps l to $h-1$.

This bounds the exponent of s by $1 + \cdots + l-1 + (h-l-1)(l-1) = (l-1)(h-l/2-1)$. All together we obtain the bound $(l-1)(w+h-1-l/2-1)$ which is strictly smaller than $(l-1)(d-l/2-1)$.

Case 2 ($h < l$): The degree of s inside

$$\sum_{i_1=1}^{s\lambda_1+l} \cdots \sum_{i_{l-1}=1}^{s\lambda_{l-1}+2} \sum_{\pi \in S_l} \text{sgn}(\pi) Q_{\alpha_0}(sk, (s\lambda)_\pi - (i_1 \dots i_{l-1}))$$

is bounded by the degree of s in the total sum of all multiplicities in the decomposition of $(S^{sk})^{\otimes h}$. We could now proceed as above, but [FZ15, Theorem 1.2(ii)] directly gives that this degree equals $\binom{h}{2}$, and hence, the total degree in which s can appear is at most

$$(w-1)(l-1) + \binom{h}{2}.$$

After using $\binom{h}{2} \leq (l-1)(h-1)/2$ this is seen as strictly smaller than $(l-1)(d-l/2-1)$. \square

Theorem 4.2. *Fix a partition μ of d . The multiplicity of the isotypic component corresponding to λ inside $S^\mu(S^k(V))$ is a piecewise quasi-polynomial in k and λ . In each full-dimensional conical chamber, its highest degree term equals $\frac{\dim \mu}{d!}$ times the highest degree term of the multiplicity of λ in $S^k(V)^{\otimes d}$.*

Proof. Let $\alpha \neq (1, \dots, 1)$ and suppose that $\sum_{\pi} \text{sgn}(\pi) Q_{\alpha}(k, \lambda)$ has a leading term of degree greater than or equal to $(l-1)(d-l/2-1)$. Pick a λ where the leading term does not vanish. Using this λ in Lemma 4.1 yields a contradiction. Now the result follows since for regular λ the contribution from $\alpha = (1, \dots, 1)$ is of degree $(l-1)(d-l/2-1)$, and each full-dimensional conical chamber contains a regular λ . \square

Conjecture 4.3. *Let μ be a partition of d and $\lambda = (\lambda_1^{a_1}, \dots, \lambda_q^{a_q})$ be a partition of kd with $l = \sum_{j=1}^q a_j$ parts. The multiplicity of $s\lambda$ in $S^\mu(S^{sk})$ is a quasi-polynomial in s whose lead term has degree*

$$\dim P_{k,d}^{\lambda} = 1 + \dots + (l-1) + (l-1)(d-l) - \sum_{j=1}^q \binom{a_j}{2} = (l-1)(d-l/2-1) - \sum_{j=1}^q \binom{a_j}{2},$$

and coefficient $\frac{\dim \mu}{d!} \text{Vol } P_{k,d}$.

Note that the degree in the conjecture is an obvious upper bound and that Theorem 4.2 yields the conjecture whenever λ is a regular partition, i.e., when λ belongs to the interior of the cone of valid parameters.

Remark 4.4. An interesting question asked by Mulmuley is whether counting functions on individual rays are Ehrhart functions of some rational polytopes. In some cases, we can provide a negative answer using reciprocity (cf. [KW, BOR09b]). For the details, see [KM15].

5. APPENDIX

5.1. Vector partition functions. Consider a polyhedral cone in standard representation $\mathcal{C} = \{Ax \geq b\} \subset \mathbb{Q}^N$, and a linear map $\pi : \mathcal{C} \rightarrow \mathbb{Q}^n$. The image of π is a polyhedral cone denoted \mathcal{D} . In this situation, the preimage of an integral point in \mathcal{D} is a polyhedron and we are interested in the (possibly infinite) number of integral points it contains. The *counting function* is

$$\begin{aligned} \phi : \mathcal{D} \cap \mathbb{N}^n &\rightarrow \mathbb{N}_0 \cup \{\infty\} \\ \phi(d) &= \#\{c \in \mathcal{C} \cap \mathbb{N}^N : \pi(c) = d\} \end{aligned}$$

The preimage of any rational point $d \in \mathcal{D}$ under π is a polyhedron and there are only finitely many combinatorial types of polyhedra appearing among all preimages (see [VSB⁺04] for an overview on the history of this result with a focus on implementation). The type depends on which supporting hyperplanes of \mathcal{C} intersect a given preimage $\pi^{(-1)}(d)$ nontrivially. This

yields a decomposition of \mathcal{D} known as the *chamber decomposition*. As in each chamber the combinatorial type of each fiber is the same, the counting function is a quasi-polynomial since in general the fiber is a rational polytope. All-together, ϕ is a piecewise quasi-polynomial.

In full generality Sturmfels has shown that the lattice point enumerator of a parametric polyhedron $\{x : Ax \leq b(t)\}$ is a piecewise quasi-polynomial in the parameters t , whenever $b(t) \in \mathbb{Z}[t]$ is a linear polynomial [Stu95]. He calls ϕ the *vector partition function* as it counts the number of ways to write a vector in terms of generators of \mathcal{C} .

Example 5.1. Fix $d \in \mathcal{D}$ and consider points kd , $k \in \mathbb{N}$ on the ray generated by d . In this case $\phi(kd)$ equals $P_{\pi^{-1}(d)}(k)$, the Ehrhart quasi-polynomial of $\pi^{-1}(d)$. If $\pi^{-1}(d)$ happens to be an integral polytope, then so are the polytopes $\pi^{-1}(kd)$ and in this case $P_{\pi^{-1}(d)}$ is an honest polynomial [Ehr77].

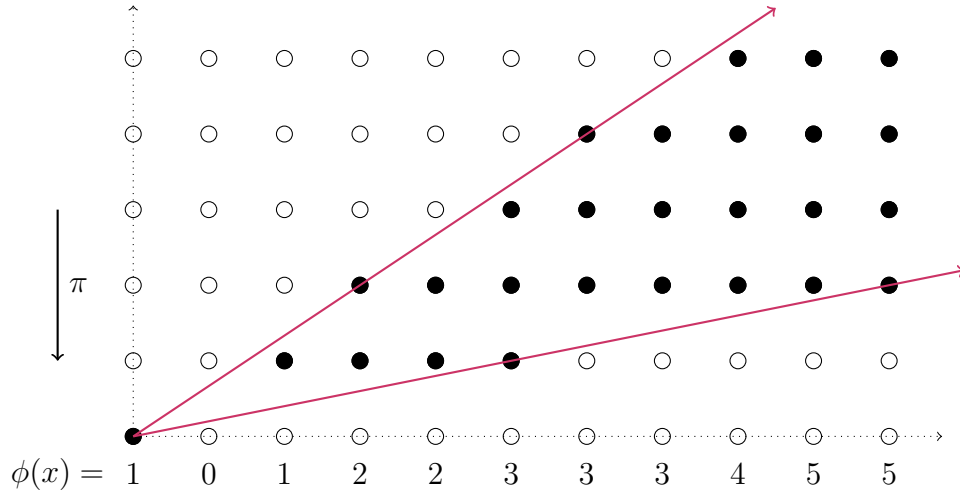


FIGURE 1. The counting problem in Example 5.2.

Example 5.2. Consider the two-dimensional cone \mathcal{C} over the matrix $\begin{pmatrix} 5 & 3 \\ 1 & 2 \end{pmatrix}$, depicted in Figure 1. Let π be the projection to the first coordinate such that the image cone \mathcal{D} is just the x -axis. The vector partition function counting the number of integer points in a vertical slice of \mathcal{C} is the piecewise quasi-polynomial

$$\phi(x) = \begin{cases} 0 & x < 0, \\ (x+1) - \lfloor \frac{x+2}{3} \rfloor - \lfloor \frac{x+4}{5} \rfloor & x \geq 0. \end{cases}$$

Note that in general a quasi-polynomial can be written as a polynomial expression in the variables and floor functions of linear functions in the variables.

Vector partition functions can be computed symbolically. The first step is usually to compute the chamber decomposition for which several algorithms exist. Once the problem is reduced to determining a quasi-polynomial for each chamber, interpolation may be the first idea that comes to mind. This is known as Clauss' method and was indeed the first method suggested for the determinations of vector partition functions. This method has problems since it can be difficult to find sufficiently many lattice points for interpolation. A more efficient approach is to use Barvinok's method which, in the setting of vector partition functions, was first suggested by Verdoolaege et al. [VSB⁺07]. Their software BARVINOK

(together with the ISL library) is the most advanced tool available today. The software is very well developed because of its applications in computer science, for instance to loop optimization in compiler development. It is one of the mathematical software tools with which run time is short compared to the time that humans need to learn something from the result. The introduction of [VBBC05] contains many references.

5.2. Computation of the plethysm coefficient quasi-polynomials. As a proof of concept we evaluated equation (3.1) using BARVINOK. We describe the necessary steps for $d = 5$ here. The input files for $d = 3, 4$ are also available on our project homepage.

In (3.1) the innermost evaluation is the quasi-polynomial function Q_α which depends on α (a partition of d) and a permutation π . Q_α is a function in λ , but through the series of reductions in Section 3, the final formula has different arguments, called b_1, \dots, b_{d-1}, s (see Remark 3.8). By convention, the b_i are ordered increasingly with b_1 the smallest.

Example 5.3. Suppose we want to determine the multiplicity of the isotypic component for $\lambda = (3, 2, 1)$ in $S^3(S^2)$. This multiplicity is equal to the multiplicity of the isotypic component of $(2, 1, 0)$ in $\bigwedge^3(S^1)$ (Remark 3.8). To evaluate it using our programs, we plug $(b, s) = (1, 1)$ into the quasi-polynomial stored in `111.qpoly`. We find that the multiplicity is equal to 0. See the last item in Section 5.4 for how this 0 is presented, though.

In the following we describe the steps necessary to repeat our computations and determine the quasi-polynomials in (3.1). The procedure consists of roughly four steps:

- (1) Determine Q_α enumerators with `barvinok_enumerate`.
- (2) Sum over the partition α using precomputed coefficients $\chi_\mu(\alpha)D_\alpha$.
- (3) Sum over permutations π .
- (4) Division by $\pm d!$ and postprocessing.

5.2.1. Enumeration. The directory `barvinok_enumeration` contains input files for the program `barvinok_enumerate`, corresponding to determination of the Q_α for different values of α and shifted λ . These files need to be processed individually with the command

```
barvinok_enumerate --to-isl < "${input}" > "${input}.result
```

where `${input}` is a filename of one of the `.barv` files. We provide the bash script `do.sh` which runs on four parallel processors for this job. In our experience, this step, for $d = 5$, should complete in less than 12 hours even on laptop computers. At this point one could argue that we are doing a lot of computation that is not strictly necessary, since the $Q_\alpha(k, \lambda_\pi)$ for different π are the same quasi-polynomials, evaluated at slightly shifted arguments. In principle one would like to compute one quasi-polynomial and evaluate it on shifted arguments. This point is legit, but it seemed more convenient using recomputation with modified constraints. Due to the parallelization, computing all $Q_\alpha(k, \lambda_\pi)$ individually by modifying the input was quick. The result of this computation are stored in `.result` files which we need for the next, computationally more demanding, step: summation.

5.2.2. Summation. At this point the `.result` files should contain all lattice point enumerators $Q_\alpha(k, \lambda_\pi)$ and our next task is to sum them with appropriate coefficients. After some experimentation it turned out to be advantageous to first sum over the partition α and later over the permutation π . In the light of Remark 3.11 the reason for this seems to be the creation of many more small chambers during summation over π . For this step we use ISCC, an interactive (and scriptable) ISL frontend which is distributed with BARVINOK. Place

the result files in the summation folder which already contains the appropriate summation scripts. Their names are `sum11111.iscc` for $\mu = (11111)$ and so on. These scripts run for a while. Here are some approximate run times that we measured on an Intel Core i7-4770 (3.4GHz):

script	runtime in hours
<code>sum11111.iscc</code>	118
<code>sum2111.iscc</code>	38
<code>sum221.iscc</code>	7
<code>sum311.iscc</code>	2
<code>sum32.iscc</code>	7
<code>sum41.iscc</code>	38
<code>sum5.iscc</code>	118

In principle this computation could be parallelized too by structuring summation hierarchically in the form of a tree. At the moment ISCC has no native support for parallelism so the only way to parallelize this computation would be on the OS level. This in turn means that intermediate results have to be written to disk and read again. Reading large quasi-polynomials is very slow (see Section 5.4), and consequently we ran each summation on one thread, but different summations at the same time.

The script `sumX.iscc` stores its result in `X.result`. This file then contains the quasi-polynomial we are looking for, multiplied with a signed factorial. In the case of $d = 5$, the factor is $5! = 120$.

5.2.3. Postprocessing. In the final step we divide the quasi-polynomial by the appropriate factorial and sign and use a text editor to convert the results from parametric sets of constant functions into functions (see Section 5.4).

5.3. Experiences and limitations. The results of our computation are piecewise quasi-polynomials and their representations are far from unique. The most basic phenomenon bothering us is that divisibility conditions may be obfuscated by existential quantifiers. For one example, if s is a variable, then $s \equiv 0 \pmod{5}$ may appear as $\exists e_0 = \lfloor (-1 + s)/5 \rfloor, \exists e_1 = \lfloor s/5 \rfloor$ such that $5e_1 = s$ and $5e_0 \leq -2 + s$ and $5e_0 \geq -5 + s$. To see the equivalence, note that the e_0 condition is that s leaves any remainder except 1 modulo 5, and thus redundant. At the moment there seems to be no automatic way to remove such redundant conditions while they do appear frequently (this one is taken from Example 5.6).

Another challenge to be addressed in the future is the number of chambers that appears after doing arithmetic with quasi-polynomials. In principle there can be chambers C_1, C_2 with corresponding quasi-polynomials p_1, p_2 such that C_2 is a face of C_1 , and p_1 when restricted to C_2 equals p_2 . At the moment BARVINOK has no means to detect this case during the computation, or rectify it a posteriori. We can not precisely estimate how much this effect hits us. We did run ISCC's `coalesce` function on each of our results which uses simple tests to detect empty chambers. This has reduced the output of the summation part to approximately a fifth of its original size.

5.4. Quirks. Using BARVINOK and ISCC, the following things occurred to us:

- It can take very long to read quasi-polynomials from disk. Our largest result files are `11111.qpoly` and `5.qpoly` which on a Core i7-4770 (3.4GHz) needed 5 hours and 51 minutes to be parsed. In contrast they need only a second to be written to disk! We

asked on the ISL development mailing list and it was confirmed that the parser is not very efficient.

- Reading a quasi-polynomial and then writing it out again need not yield the same representation. The parser that is used to read piecewise quasi-polynomials from files applies certain transformations that are not applied when computing the quasi-polynomials from scratch.
- Mathematically speaking our results are simply functions $\mathbb{N}^d \rightarrow \mathbb{N}$, but in the computer things are not that simple. The program `barvinok_enumerate` which we use as the first step in our computation does not return functions on \mathbb{N}^d —it returns sets of constant functions, parametrized over \mathbb{N}^d . It is technically impossible to “evaluate” these parametric sets of constants, because only functions can be evaluated in ISL. To fix this we simply used text editing on the output files to convert expressions like

$$[b1, s] \rightarrow \{ [] \rightarrow (1/2 * b1 + 1/2 * b1^2) : \dots \}$$
into

$$\{ [b1, s] \rightarrow (1/2 * b1 + 1/2 * b1^2) : \dots \}$$
- If the result of a quasi-polynomial evaluation is a nonzero integer n , then the result is formatted as $\{n\}$. If, however, the result is zero, the empty set is returned: $\{ \}$.

5.5. Evaluation. Evaluation of explicit plethysm coefficients can be done in LiE [vLCL92] and other packages like SAGE:

Example 5.4. To evaluate plethysm in SAGE [S⁺14], first one sets up the ring of symmetric functions in the Schur basis with

```
sage: s = SymmetricFunctions(QQ).schur()
```

After this the (Schur function) plethysm can be computed by plugging in as follows:

```
sage : s([2,1,1])(s[3,1])
```

For both parametric partial and complete evaluation of our stored results, the most practical tool is ISCC.

Example 5.5. In ISCC, to evaluate a quasi-polynomial P (created, for instance with

```
P := read "111.qpoly";
```

at arguments (3,2) use the following input to ISCC:

```
P ({[3,2]});
```

The $()$ brackets are used to trigger evaluation on an isl domain introduced with $\{ \}$ which in turn consists of only one isolated point $[3, 2]$.

Example 5.6. In this example we explain how to arrive at the at the result in Example 1.3 using the provided result files. Note that by means of the reductions in Remark 3.8, this formula could in principle also be derived from the Cayley-Sylvester formula in $SL_2(\mathbb{C})$ representation theory, but this formula is not as explicit as ours. It involves counting tableaux under side constraints.

Let again $\mu = (5)$, and $\lambda = (31, 3, 2, 2, 2)$. The following code loads the quasi-polynomial for μ from the file `5.qpoly` and evaluates it along the line $s\lambda$ for $s \in \mathbb{Z}$. Note that the read command will take very long (up to several hours) since the parser for quasi-polynomials is not very optimized.

```
P:= read "5.qpoly";
{[s] -> [0,0,s, 6*s]} . P;
```

The result looks like this:

```
$2 := { [s] -> ((((((3/5 - 289/720 * s + 1/20 * s^2 + 1/720 * s^3) + (5/8 + 1/8 * s) *
floor((s)/2)) + (1/3 - 1/6 * s) * floor((s)/3)) + ((7/12 - 1/3 * s) + 1/2 *
floor((s)/3)) * floor((1 + s)/3) + 1/4 * floor((1 + s)/3)^2 + 1/4 * floor((s)/4))
- 1/4 * floor((3 + s)/4)) : exists (e0 = floor((-1 + s)/5): 5e0 = -1 + s and s >= 1);
[s] -> ((((((1 - 289/720 * s + 1/20 * s^2 + 1/720 * s^3) + (5/8 + 1/8 * s) *
floor((s)/2)) + (1/3 - 1/6 * s) * floor((s)/3)) + ((7/12 - 1/3 * s) + 1/2 *
floor((s)/3)) * floor((1 + s)/3) + 1/4 * floor((1 + s)/3)^2 + 1/4 * floor((s)/4)) -
1/4 * floor((3 + s)/4)) : exists (e0 = floor((-1 + s)/5), e1 = floor((s)/5): 5e1 = s
and s >= 5 and 5e0 <= -2 + s and 5e0 >= -5 + s); [s] -> ((((((((-4/5 + 289/720 * s -
1/20 * s^2 - 1/720 * s^3) + (-5/8 - 1/8 * s) * floor((s)/2)) + (-1/3 + 1/6 * s) *
floor((s)/3)) + ((-7/12 + 1/3 * s) - 1/2 * floor((s)/3)) * floor((1 + s)/3) - 1/4 *
floor((1 + s)/3)^2 - 1/4 * floor((s)/4)) + 1/4 * floor((3 + s)/4)) * floor((s)/5) +
((((((4/5 - 289/720 * s + 1/20 * s^2 + 1/720 * s^3) + (5/8 + 1/8 * s) * floor((s)/2)) +
(1/3 - 1/6 * s) * floor((s)/3)) + ((7/12 - 1/3 * s) + 1/2 * floor((s)/3)) *
floor((1 + s)/3) + 1/4 * floor((1 + s)/3)^2 + 1/4 * floor((s)/4)) - 1/4 *
floor((3 + s)/4)) * floor((3 + s)/5)) : exists (e0 = floor((-1 + s)/5), e1 =
floor((s)/5): s >= 1 and 5e0 <= -2 + s and 5e0 >= -5 + s and 5e1 <= -1 + s and
5e1 >= -4 + s); [s] -> 1 : s = 0 }
```

To parse this, first observe that a new chamber starts whenever we see `[s] ->`. The first step towards understanding this output is to isolate the four chambers and to reformulate their constraints. The constraints are the items after the colon in each chamber.

Chamber 1

```
exists (e_0 = floor((-1 + s)/5): 5e0 = -1 + s and s >= 1)
```

which means $s \geq 1$ and $s \equiv 1 \pmod{5}$.

Chamber 2

```
exists (e0 = floor((-1 + s)/5), e1 = floor((s)/5):
5e1 = s and s >= 5 and 5e0 <= -2 + s and 5e0 >= -5 + s)
```

which, except from $s \geq 5$, translates into the requirement that s should leave remainder zero modulo 5, and additionally $s - 5 \leq 5\lfloor \frac{s-1}{5} \rfloor \leq s - 2$. The second condition is that s leaves any remainder except 1 modulo 5, and thus redundant. At the moment our computational tools are unable to carry out this simplification automatically.

Chamber 3

```
exists (e0 = floor((-1 + s)/5), e1 = floor((s)/5): s >= 1 and 5e0
<= -2 + s and 5e0 >= -5 + s and 5e1 <= -1 + s and 5e1 >= -4 + s).
```

The conditions are $s \geq 1$, $s - 5 \leq 5\lfloor \frac{s-1}{5} \rfloor \leq s - 2$, and $s - 4 \leq 5\lfloor \frac{s}{5} \rfloor \leq s - 1$. They are both satisfied if and only if s leaves remainder 2, 3, or 4 modulo 5.

Chamber 4

This chamber is singleton: $s = 0$ and thus the case distinction is complete.

The output of our program has each quasi-polynomial written in an expression involving floor functions. To simplify the presentation, let us introduce the following shorthands which

appear in the output

$$\begin{aligned}
p_1 &= \frac{3}{5} - \frac{289}{720}s + \frac{1}{20}s^2 + \frac{1}{720}s^3, \\
p_2 &= \frac{5}{8} + \frac{1}{8}s, \quad p_3 = \frac{1}{3} - \frac{1}{6}s, \quad p_4 = \frac{7}{12} - \frac{1}{3}s, \\
q_1 &= 1 - \frac{289}{720}s + \frac{1}{20}s^2 + \frac{1}{720}s^3, \\
r_1 &= \frac{4}{5} - \frac{289}{720}s + \frac{1}{20}s^2 + \frac{1}{720}s^3.
\end{aligned}$$

Using these shorthands and only trivial manipulations of the output we arrive at the following three quasi-polynomials in the three nontrivial chambers:

$$\begin{aligned}
&\underline{s \equiv 1 \pmod{5}} \\
&p_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left(p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left\lfloor \frac{1+s}{3} \right\rfloor + \frac{1}{4} \left(\left\lfloor \frac{1+s}{3} \right\rfloor^2 + \left\lfloor \frac{s}{4} \right\rfloor - \left\lfloor \frac{3+s}{4} \right\rfloor \right) \\
&\underline{s \equiv 0 \pmod{5}} \\
&q_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left(p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left\lfloor \frac{1+s}{3} \right\rfloor + \frac{1}{4} \left(\left\lfloor \frac{1+s}{3} \right\rfloor^2 + \left\lfloor \frac{s}{4} \right\rfloor - \left\lfloor \frac{3+s}{4} \right\rfloor \right) \\
&\underline{s \equiv 2, 3, 4 \pmod{5}} \\
&r_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left(p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left\lfloor \frac{1+s}{3} \right\rfloor + \frac{1}{4} \left(\left\lfloor \frac{1+s}{3} \right\rfloor^2 + \left\lfloor \frac{s}{4} \right\rfloor - \left\lfloor \frac{3+s}{4} \right\rfloor \right)
\end{aligned}$$

There is an obvious pattern here, but unfortunately the ISL engine has problems with factoring out, or simplifying these expressions automatically. For instance in the third chamber it actually returns the expression

$$\begin{aligned}
&\underline{s \equiv 2, 3, 4 \pmod{5}} \\
&\left(r_1 - p_2 \left\lfloor \frac{s}{2} \right\rfloor - p_3 \left\lfloor \frac{s}{3} \right\rfloor - \left(p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left\lfloor \frac{1+s}{3} \right\rfloor - \frac{1}{4} \left\lfloor \frac{1+s}{3} \right\rfloor^2 - \frac{1}{4} \left\lfloor \frac{s}{4} \right\rfloor + \frac{1}{4} \left\lfloor \frac{3+s}{4} \right\rfloor \right) \left\lfloor \frac{s}{5} \right\rfloor + \\
&\left(-r_1 + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left(p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left\lfloor \frac{1+s}{3} \right\rfloor + \frac{1}{4} \left\lfloor \frac{1+s}{3} \right\rfloor^2 + \frac{1}{4} \left\lfloor \frac{s}{4} \right\rfloor - \frac{1}{4} \left\lfloor \frac{3+s}{4} \right\rfloor \right) \left\lfloor \frac{3+s}{5} \right\rfloor.
\end{aligned}$$

Not only can we simplify the presentation, in fact the above expression looks like the lead term would be of quasi-polynomial nature while in reality it is not since

$$\left(\left\lfloor \frac{s}{5} \right\rfloor - \left\lfloor \frac{3+s}{5} \right\rfloor \right) = -1 \quad \text{if } s \equiv 2, 3, 4 \pmod{5}.$$

Applying all simplifications and the shortcuts

$$p = -\frac{289}{720}s + \frac{1}{20}s^2 + \frac{1}{720}s^3,$$

$$p_2 = \frac{5}{8} + \frac{1}{8}s, \quad p_3 = \frac{1}{3} - \frac{1}{6}s, \quad p_4 = \frac{7}{12} - \frac{1}{3}s,$$

$$A(s) = p + p_2 \left\lfloor \frac{s}{2} \right\rfloor + p_3 \left\lfloor \frac{s}{3} \right\rfloor + \left(p_4 + \frac{1}{2} \left\lfloor \frac{s}{3} \right\rfloor \right) \left\lfloor \frac{1+s}{3} \right\rfloor + \frac{1}{4} \left(\left\lfloor \frac{1+s}{3} \right\rfloor^2 + \left\lfloor \frac{s}{4} \right\rfloor - \left\lfloor \frac{3+s}{4} \right\rfloor \right),$$

the final result is

$$Q(s) = A(s) + \begin{cases} 1 & \text{if } s \equiv 0 \pmod{5} \\ \frac{3}{5} & \text{if } s \equiv 1 \pmod{5} \\ \frac{4}{5} & \text{if } s \equiv 2, 3, 4 \pmod{5}. \end{cases}$$

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